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COMMENT

A comment on 'On the multifractal nature of fully developed turbulence and chaotic systems'

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Abstract. A distinction is drawn between construction rules leading to inhomogeneous and homogeneous random fractal configurations.

In the work of Benzi *et al* (1984), henceforth referred to as BPPV, results are given for the higher moments of random fractal ensembles. The purpose of this comment is to make clear the distinction between results for higher moments in case (a) in which at each step the decoration for each eddy is chosen independently from some distribution and case (b) in which at each step some decoration is chosen from the distribution and then used to decorate all eddies; the separation of these two cases is also made in Mandelbrot *et al* (1985).

The most powerful tool for discussing fractal ensembles is the partition or generating function (Melrose 1986). Let L stand for the number of eddies in some configuration or some decoration and let $Z_1(S)$ be the partition function for the basic set of decorations; in the case of the distribution (3.9) of BPPV $Z_1(S)$ is

$$Z_1(S) = R(S) = xS^2 + (1-x)S^4, \tag{1}$$

where $x/6$ is the probability of decorating some eddy by one of the six possible decorations with two sub-eddies.

Now in case (a), which seems to be that discussed in § 3 of BPPV, the ensemble is hierarchical (Melrose 1986): the partition function for the ensemble of configurations possible at the nth decoration obeys

$$Z_n(S) = Z_{n-1}(R(S)) \tag{2}$$

and can be found by an iterative substitution starting from (1). The statistics of the self-similar random fractal can be calculated from Z_n at the fixed point of (1): $S^* = 1$. Now whilst the expected number of eddies on an nth level configuration, $\langle L \rangle_n$, obeys

$$\langle L \rangle_n = \frac{(S\, dZ_n(S)/dS)|_{S=1}}{Z_n(1)} = (dR(S)/dS|_{S=1})^n = \lambda_1^n, \tag{3}$$

as given in BPPV equation (3.4), the second moment, $\langle L^2 \rangle_n$, obeys

$$\begin{aligned} \langle L^2 \rangle_n &= \langle L \rangle_n + \frac{(S^2\, d^2 Z_n(S)/dS^2)|_{S=1}}{Z_n(1)} \\ &= \lambda_1^n + \lambda_1^{2n}\lambda_2(1-\lambda_1^{-n})/(\lambda_1^2 - \lambda_1) \end{aligned} \tag{4}$$

where $\lambda_2 = (\mathrm{d}^2 R(S)/\mathrm{d}S^2|_{S=1})$; see Melrose (1986) for derivations of (3) and (4). The result (4) is contrary to that implied by equation (3.8) of BPPV:

$$\langle L^q\rangle_n = \langle L^q\rangle_1^n. \tag{5}$$

As shown below this result only holds in case (b). From (4) one finds that in fact

$$\langle L^2\rangle_n = K_n(b^n)^{2D} \tag{6}$$

where $K_n \to$ constant as $n\to\infty$, b^n is the Euclidean linear scale of the nth level configurations and $D=\log(\lambda_1)/\log(b)$ is the fractal dimension of the ensemble. Note that (6) suggests that the random self-similar ensemble generated under case (a) does obey a statistical homogeneity in the sense of BPPV equation (4.5).

The case (b) which unambiguously is that described in § 4 of BPPV is now discussed. The partition function is not hierarchical, if for the basic set of decorations

$$Z_1(S) = \sum_L p_L S^L, \tag{7}$$

one finds for all eddies decorated the same at each step

$$Z_n(S) = \sum_{\{L_1 L_2 \ldots L_n\}} p_{L_1} p_{L_2} \ldots p_{L_n} S^{(L_1 L_2 \ldots L_n)}, \tag{8}$$

where the L_i are summed over the range of L in (7). Statistics of the ensemble are calculated at $S=1$ and directly one finds the result (5):

$$\langle L^q\rangle_n = \sum_{\{L_1 L_2 \ldots L_n\}} (L_1 L_2 \ldots L_n)^q p_L p_{L_2} \ldots p_{L_n} = \left(\sum_L L p_L\right)^q = \langle L^q\rangle_1^n, \tag{9}$$

as given by BPPV equation (4.9).

In summary whilst in case (b) expectations at each step can be taken independently (there being only one selection from the distribution at each step), in case (a) there is a subtle coupling between steps in that the number of independent expectations to be taken at step m depends on the expected number of eddies (the first moment) at step $m-1$; the result (4) shows that this effect dominates the statistics.

The above results have significance physically. The construction (b) would seem to be unphysical in that it possesses an implicit global correlation surely not to be expected in random systems. Given that the more physical fractal ensemble constructed under (a) does possess a statistical homogeneity (higher moments related to the first moment by power laws) deviations from homogeneity observed in actual systems can not be explained just by a randomisation of the fractal model. What sort of physically reasonable model need be invoked to explain a breakdown in homogeneity is a good topic for future investigation.

References

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